Principal Component Analysis of the Changes of the Interest Rate Curve

Let $\mathbf{r} = (r_1, ..., r_m)$, where r_i is the interest rate with maturity t_i, $i = 1, ..., m$. These rates change with time. Let $\partial\mathbf{r} = \mathbf{r}^{''} - \mathbf{r}^{'}$ denote the change between two consecutive days. We shall consider these changes as outcomes of a random variable with covariance matrix $Q = (q_{i,j})$.

This matrix is thus symmetric and positive definite. It has m pairwise orthogonal eigenvectors $\mathbf{a}_1, ..., \mathbf{a}_m$ of length 1, $\mathbf{a}_i \cdot \mathbf{a}_k = 0$ for $i \neq k$, $\mathbf{a}_i \cdot \mathbf{a}_i = 1$, because it is symmetric. All eigenvalues are positive because the matrix is positive definite. Write σ_i^2 for the eigenvalue corresponding to the eigenvector $\mathbf{a}_i$, and assume that they are numbered in decreasing order, $\sigma_1 \geq ... \geq \sigma_m$.

Put $\mathbf{X} = \partial\mathbf{r} - E[\partial\mathbf{r}]$. Let $\xi_1, ..., \xi_m$ be the coordinates of $\mathbf{X}$ in the basis $\mathbf{a}_1, ..., \mathbf{a}_m$,

$$\mathbf{X} = \xi_1\mathbf{a}_1 + ... + \xi_m\mathbf{a}_m.$$

The random variables $\xi_1, ..., \xi_m$ satisfies $\xi_i = \mathbf{a}_i \cdot \mathbf{X}$, and hence have expectation 0 and covariances

$$E[\xi_i\xi_j] = E[(\mathbf{a}_i \cdot \mathbf{X})(\mathbf{a}_j \cdot \mathbf{X})] = E[\sum_{k,l} \mathbf{a}_i(k)\mathbf{X}(k)\mathbf{a}_j(l)\mathbf{X}(l)] =$$

$$\sum_{k,l} \mathbf{a}_i(k)q_{k,l}\mathbf{a}_j(l) = \mathbf{a}_i \cdot Q\mathbf{a}_j = \sigma_j^2\mathbf{a}_i \cdot \mathbf{a}_j.$$

They are therefore uncorrelated and ξ_i has variance σ_i^2.

We have thus

$$\partial\mathbf{r} = E[\partial\mathbf{r}] + \xi_1\mathbf{a}_1 + ... + \xi_m\mathbf{a}_m.$$

The expected value $E[\partial\mathbf{r}]$ is negligible compared to the standard deviations σ_i. Therefore

$$\partial\mathbf{r} \approx \xi_1\mathbf{a}_1 + ... + \xi_m\mathbf{a}_m.$$