SUPPLEMENT TO SECTION 8.1

Portfolios with maximal Sharpe ratio

We shall here give the following result.

Proposition S8.1 *Consider the portfolio that has the stock weights $\boldsymbol{v}$, and the remainder in cash.*

(a) Show that the Sharpe ratio of the porfolio equals

$$\frac{\boldsymbol{v} \cdot (\boldsymbol{\mu} - r_f \mathbf{1})}{\sqrt{\boldsymbol{v} \cdot Q\boldsymbol{v}}}$$

regardless the amount of cash.

(b) Show that $\boldsymbol{v}$ maximizes the Sharpe ratio if and only if $\boldsymbol{v} = c\boldsymbol{v}_{max}$ for some $c > 0$, and that the maximal Sharpe ratio equals σ_{max}.

Proof.

(a) The Sharpe ratio equals

$$\frac{(1-\omega)r_f + \boldsymbol{v}\cdot\boldsymbol{\mu} - r_f}{\sqrt{\boldsymbol{v}\cdot Q\boldsymbol{v}}} = \frac{\boldsymbol{v}\cdot(\boldsymbol{\mu} - r_f\mathbf{1})}{\sqrt{\boldsymbol{v}\cdot Q\boldsymbol{v}}}$$

because $\boldsymbol{v}\cdot\mathbf{1} = \omega$.

(b) First maximize the Sharpe ratio under the constraint that the portfolio volatility equals σ, and then maximize over σ. The first maximum is according to (6.34) attained for $\boldsymbol{v} = c\boldsymbol{v}_{max}$ with $c = \sigma/\sigma_{max}$. It follows from (6.26) and (6.27) that the maximal Sharpe ratio equals σ_{max}, and hence is independent of σ. □