Supplement to Section 6.2

Maximal drift under constraints on maximal volatility and stock weight

To be read after Section 6.2.5.

We have maximized the drift (6.16) under various constraints. These constraints are identities. The relevant constraints are, however, often inequalities, and it is not always obvious which of the optimal portfolios that will give the maximum.

Let

$$\sigma_P(\mathbf{v}) = \sqrt{\mathbf{v} \cdot Q\mathbf{v}} \text{ and } w_P(\mathbf{v}) = \mathbf{1} \cdot \mathbf{v}$$

denote the portfolio volatility and the sum of the stock weights, respectively. We shall write

$$w_{max} = \mathbf{1} \cdot \mathbf{v}_{max} = (\mu_{min} - r_f)/\sigma_{min}^2$$

for the sum of the stock weights of the maximal drift portfolio. (This is denoted ω in the book.)

Recall that the following portfolios have maximal drift for given values of σ_P or w_P.

Case A. No constraint. In this case $\mathbf{v} = \mathbf{v}_{max}$, $\sigma_P(\mathbf{v}) = \sigma_{max}$. and $w_P(\mathbf{v}) = w_{max}$.

Case B. Constraint: $\sigma_P(\mathbf{v}) = \sigma$. In this case $\mathbf{v} = \sigma\mathbf{v}_{max}/\sigma_{max}$, and $w_P(\mathbf{v}) = \sigma w_{max}/\sigma_{max}$.

Case C. Constraint: $w_P(\mathbf{v}) = w$. In this case $\mathbf{v} = w\mathbf{v}_{min} + \mathbf{v}_{aux}$, and $\sigma_P(\mathbf{v})^2 = w^2\sigma_{min}^2 + \sigma_{aux}^2$.

Case D. Constraint: $\sigma_P(\mathbf{v}) = \sigma$ *and* $w_P(\mathbf{v}) = w$. In this case $\mathbf{v} = w\mathbf{v}_{min} + k\mathbf{v}_{aux}$, where $k = \frac{\sqrt{\sigma^2 - w^2\sigma_{min}^2}}{\sigma_{aux}}$.

We shall assume that $w_{max} > 0$ and maximize the drift, $\nu_P(\mathbf{v})$ for $\mathbf{v}$ on the set

$$S = \{\mathbf{v} \in \mathbb{R}^m; \sigma_P(\mathbf{v}) \leq \sigma \text{ and } w_P(\mathbf{v}) \leq w\}.$$

Here σ and w are two given positive numbers. If $w_{max} < 0$, then it may be more natural to replace the inequality $w_P(\mathbf{v}) \leq w$ in the definition of S by $w_P(\mathbf{v}) \geq w$ with $w < 0$. We will comment on this at the end of this supplement.

The set S is compact, and hence the maximum is attained on it. We shall split the boundary, ∂S, into three parts.

$$\partial S_\sigma = \{\mathbf{v} \in \mathbb{R}^m; \sigma_P(\mathbf{v}) = \sigma \text{ and } w_P(\mathbf{v}) < w\}.$$

$$\partial S_w = \{\mathbf{v} \in \mathbb{R}^m; \sigma_P(\mathbf{v}) < \sigma \text{ and } w_P(\mathbf{v}) = w\}.$$

$$\partial S_{\sigma,w} = \{\mathbf{v} \in \mathbb{R}^m; \sigma_P(\mathbf{v}) = \sigma \text{ and } w_P(\mathbf{v}) = w\}.$$

If the maximum is attained in the interior we have Case A. In this case

$$\sigma_P(\mathbf{v}) = \sigma_{max}, \text{ and } w_P(\mathbf{v}) = w_{max}.$$

Therefore this holds if

$$\sigma > \sigma_{max} \text{ and } w > w_{max}.$$

Next, consider the possibility that the maximum is attained on ∂S_σ. If so, we have Case B. In this case

$$\sigma_P(\mathbf{v}) = \sigma, \text{ and } w_P(\mathbf{v}) = \frac{\sigma}{\sigma_{max}} w_{max}.$$

Therefore this holds if

$$\frac{\sigma}{\sigma_{max}} < \frac{w}{w_{max}} \text{ and } \sigma \leq \sigma_{max}.$$

(Note that if the last inequality does not hold, then the maximum is attained in the interior.)

Another possibility is that the maximum is attained on ∂S_w. If so, we have Case C. In this case

$$\sigma_P(\mathbf{v})^2 = w^2\sigma_{min}^2 + \sigma_{aux}^2, \text{ and } w_P(\mathbf{v}) = w.$$

Therefore this holds if

$$w^2\sigma_{min}^2 + \sigma_{aux}^2 < \sigma^2 \text{ and } w \le w_{max}.$$

(Note that if the last inequality does not hold, then by (6.30) the maximum is attained in the interior.)

The remaining possibility is that the maximum is attained on $\partial S_{\sigma,w}$, and this is thus the case if

$$\sigma \le \sigma_{max}, \ \frac{\sigma}{\sigma_{max}} \ge \frac{w}{w_{max}} \text{ and } \sigma^2 \le w^2\sigma_{min}^2 + \sigma_{aux}^2,$$

and hence Portfolio D gives the maximum for these values of σ and w.

Figure 1 shows the above domains in the (σ, w)-plane with σ on the horizontal axis. The values of the parameters are $\sigma_{min} = 0.22$, $\sigma_{max} = 0.68$, $\sigma_{aux} = 0.49$ and $w_{max} = 2.13$.

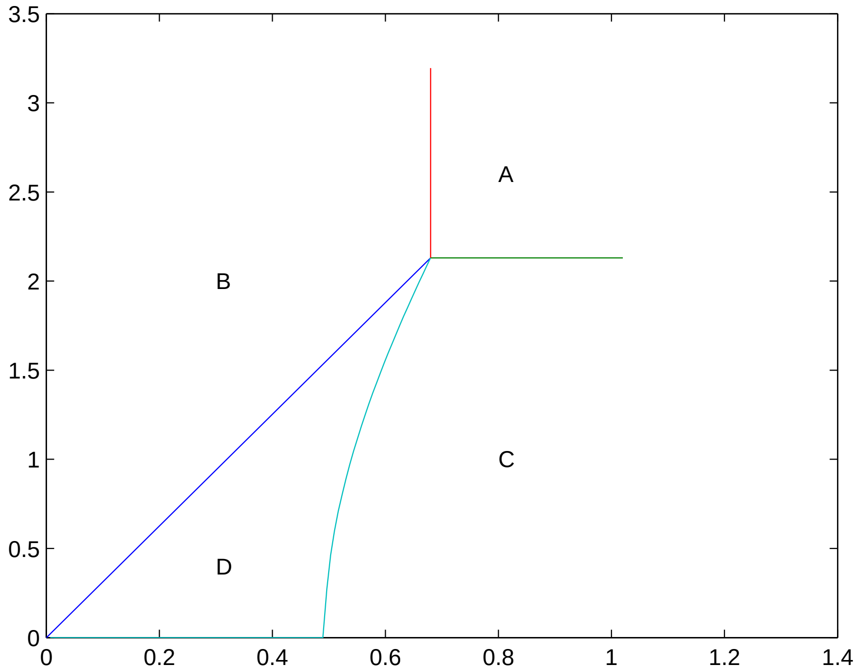

Figure 1: Domains for maximal drift portfolios.

EXERCISE. Verify directly that the weights of the portfolios on each side of the separating lines coincide on the lines.

Figure 2 shows a three dimensional plot of the maximal drift as a function of σ and w.

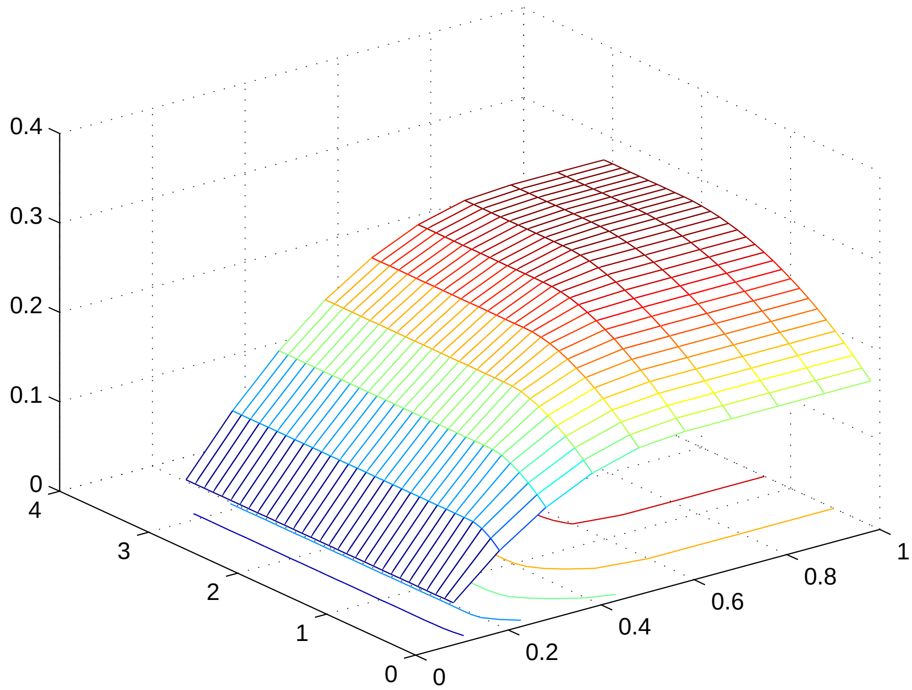

Figure 2: Maximal drift as a function of maximal volatility and maximal stock weight.

If $w_{max} < 0$, and the inequality $w_P(\mathbf{v}) \leq w$ in the definition of S is replaced by $w_P(\mathbf{v}) \geq w$ with $w < 0$, we just have to mirror the plot in Fig. 1 in the σ-axis, $(\sigma, w) \to (\sigma, -w)$, and the plot in Fig. 2 in the (σ, ν)-plane, $(\sigma, w, \nu) \to (\sigma, -w, \nu)$.

In order to be complete. What happens if $w_{max} = 0$? In this case $\mu_{\min} = r_f$, and hence $\mathbf{v}_{max} = \mathbf{v}_{aux}$, $\sigma_{max} = \sigma_{aux}$. Therefore the (σ_P, w_P) of the portfolios A-D are $(\sigma_{aux}, 0), (\sigma, 0), (\sqrt{w^2\sigma_{min} + \sigma_{aux}^2}, w)$, and (σ, w). The maximal drift under the constraints $\sigma_P \leq \sigma$ and $w_1 \leq w_P \leq w_2$ with

$w_1 < 0$, $w_2 > 0$ is therefore attained by portfolio A if $\sigma \geq \sigma_{max}$, and by portfolio B otherwise.